A Rigorous Proof of the Inflationary Spectrum

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Abstract

Fundamental formulae representing the density perturbation spectrums generated in inflationary scenarios are rigorously proved. Quantum fluctuations as initial conditions for structure generation in some inflationary era can be calculated in general scale. Thus, we could avoid the matching of the large and the small scale results at the horizon crossing epoch. This is possible because the perturbed scalar field equation in a special gauge has some remarkable nice properties. In the uniform-curvature gauge the contributions from the metric fluctuations effectively disappear in certain expansion stages, and also we can derive a simple general solution in the large scale limit. The equation reduces to the one neglecting the metric fluctuations in two main bases of inflation scenarios, namely, the exponential and power law type background expansion; this makes the quantum generation simple. The equation leads to a simple form of large scale solution valid for general scalar field potential; this makes the classical evolution simple. We also show that the same large scale integral form solution remains valid for a wide class of generalized gravity theories.

1 Introduction

An accelerated expansion stage (often termed as inflation) in the early evolution phase of our universe provides a natural setting with which we can explain the currently observed large scale structures by some causal processes occurred during the acceleration stage. Usually a scalar field is employed in a concrete model building for the acceleration stage. There exist many qualitatively successful scenarios based on its variations [1]. A further attractive point of the early acceleration phase is that the ever present microscopic quantum fluctuations (say, in the toy scalar field) can be expanded into macroscopic quantities during the acceleration phase. This tempts us to identify them with natural seeds which can be developed into the observed large scale structure in later time. The accelerated expansion itself also provides a setting of the seemingly homogeneous and isotropic observable part of the global background in which the large scale structures are imbeded. The improved picture provided by introducing the early acceleration phase is reasonably simple compared with other attempts.

In this paper we address some points concerning the fundamental formulae which have been widely used in calculating the density (scalar type mode) fluctuations generated in the early acceleration phase. We consider a minimally coupled scalar field and some representative acceleration phases realized by varying the potential of the scalar field. As a background we consider a spatially flat, homogeneous, and isotropic model universe (Friedmann-Lemaître-Robertson-Walker; FLRW). Parts of the formulae are the following:

$$\left. \frac{\Delta\mu}{\mu} \right|_{\mathrm{MDE,HC}} = \left| -\frac{2}{5} \frac{H\Delta\hat{\phi}}{\dot{\phi}} \right|_{\mathrm{UCG,SFDE,LS}}, \quad \left. \frac{\Delta\mu}{\mu} \right|_{\mathrm{RDE,SS,AMP}} = \left| -4\frac{H\Delta\hat{\phi}}{\dot{\phi}} \right|_{\mathrm{UCG,SFDE,LS}}. \tag{1}$$

where $H(t) \equiv \dot{a}/a$, $a(t)$ is the cosmic scale factor, $\mu(t)$ and $\phi(t)$ are the background energy density and the scalar field, respectively; an overdot denotes a derivative based on the background proper-time t. One may find the formulae in Eq. (1) are similar to those which appeared in [2, 3]. Let us explain the notations used in Eq. (1): the subindices indicate UCG (uniform-curvature gauge), LS (large scale), SS (small scale), HC (horizon crossing time), MDE (matter-dominated era), RDE (radiation-dominated era), SFDE (scalar field dominated era), and AMP (amplitude). The precise meanings of $\Delta\mu(k,t)$ and $\Delta\hat{\phi}(k,t)$, and the reason for taking the absolute values and hats will be explained later in Eqs. (29) and (46). We do not specify any gauge condition for $\Delta\mu(k,t)$'s; we can consider it as the power spectrum of Newtonian density fluctuations with the reason to be explained in §4; see below Eq. (37).

In an exponential inflation case, one may often find $\Delta\hat{\phi}(k,t) = H/(2\pi)$. We do have this formula as

$$\left. \Delta\hat{\phi} \right|_{\mathrm{UCG,EXP,LS}} = \left. \frac{H}{2\pi} \right|_{\mathrm{EXP}}, \tag{2}$$

where EXP indicates that the quantities are evaluated during the exponential inflation stage supported by the scalar field; in the case of power law expansion, the corresponding formula is presented in eq.(36). Eqs. (1) and (2) will be derived later.

We would like to point out that derivations of Eqs. (1) and (2) in [2, 3, 4, 5, 6, 7, 8] were not complete compared with the presentation in this paper. (We note that the main results of this paper were previously presented in [9, 10].) For correct analyses, a proper choice of the gauge (or equivalently the gauge invariant combination) is essential. Concerning Eq. (1) the previous derivation often used the concept of the exponential inflation. The exponential expansion leads to a degenerate case where even different gauge calculations give a similar result. However, our Eq. (1) is generally applicable to scalar field dominated stage with general potential. In the case of Eq. (2), previous analyses either neglected the metric perturbations by hand (which is the case of quantum field in curved spacetime ansatz) or considered the metric perturbations only in the small scale limit where the metric fluctuation is negligible again. (In hindsight we notice that in the exponential expansion case we can manage the similar result even in other gauges because the case is degenerate.) In most of the gauge choices the full equation does approach (degenerate to) Eq. (9) in the small scale limit. However, in that way, the large scale conditions in Eqs. (1) and (2) are violated; the numerical justification of the case is another point [11]. Notice that Eq. (1) is valid when $\Delta\hat{\phi}(k,t)$ is evaluated in the large scale where the spatial gradient term is negligible. In Eq. (2) we do evaluate the quantity in the large scale. In this way, we avoid ad hoc matching of the solutions of the large and small scale limits at the horizon crossing epoch during the acceleration phase. In [12] we showed that our uniform-curvature gauge analysis is more straightforward than the zero-shear gauge analyses used in [6, 7, 8]. The derivation of quantum fluctuations in the zero-shear gauge is difficult and is currently unknown in the non-exponential case. Later we will show that, in the uniform-curvature gauge, because of a cancellation even in the case of power law expansion we can derive corresponding quantum fluctuations in an exact form; see Eq. (36).

Combining Eqs. (1) and (2) we have

$$\frac{\Delta\mu}{\mu}\bigg|_{\text{MDE,HC}} = \bigg|-\frac{1}{5\pi}\frac{H^2}{\dot{\phi}}\bigg|\bigg|_{\text{EXP}}. \tag{3}$$

(The case for radiation dominated era is similar.) In Eq. (3) the gauge conditions disappear; though we have considered the full perturbed metric contributions in the derivation. This must be true in any complete analysis. In other words, the complete analysis using any other gauge ought to lead to the same result as in Eq. (3); but not Eqs. (1) and (2) separately (separately, they are valid only in the gauge choice mentioned).

In the relativistic perturbation analyses, choosing a proper gauge condition is very important. In the analyses involving the scalar field the uniform-curvature gauge is remarkably convenient; the uniform-curvature gauge was first introduced in [13]. In the later evolution stage involving the classical ideal fluid, previously we found that using some combinations of other gauge conditions is essential for proper analyses; for example, the perturbed potential and perturbed velocity in the zero-shear gauge, and perturbed density in the comoving gauge are closely related with the corresponding Newtonian variables, and have correct Newtonian behavior in the Newtonian limit [14, 15]. Each of gauge condition we mention completely fixes the gauge mode. Thus, any variable under such gauge conditions uniquely corresponds to a gauge invariant combination. Without losing generality we can regard the variables under such gauge conditions as gauge invariant ones.

As a unit, we set $c \equiv 1$.

1.1   FLRW Background

The dynamics of a flat FLRW spacetime is governed by:

$$H^2 = \frac{8\pi G}{3}\mu, \quad \dot{H} = -4\pi G\left(\mu + p\right), \tag{4}$$

where μ and p are the energy density and the pressure, respectively. The minimally couled scalar field, ϕ, contributes to the fluid quantities as:

$$\mu = \frac{1}{2}\dot{\phi}^2 + V, \quad p = \frac{1}{2}\dot{\phi}^2 - V, \tag{5}$$

where $V(\phi)$ is the potential of the scalar field. From Eqs. (4) and (5) we can derive

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0, \tag{6}$$

where $V_{,\phi} \equiv \partial V/(\partial\phi)$.

For an exact de Sitter space, $a \equiv a_0 e^{H(t-t_0)}$ with $H =$ constant, supported by the scalar field, Eqs. (4) and (5) imply $\dot{\phi} = 0 = V_{,\phi}$. However, as long as the background dynamics is not much affected, we may allow $V(\phi)$ with nonvanishing $\dot{\phi}$ while the background evolution, H, is still approximated as de Sitter space. We consider this as the case for exponential expansion stage.

For $a \propto t^{2/(3+3w)}$ with $w \equiv p/\mu =$ constant supported by the dynamics of the background scalar field, we have:

$$\phi = \sqrt{\frac{1}{6\pi G(1+w)}} \ln t, \quad V = \frac{(1-w)}{12\pi G(1+w)^2} e^{-\sqrt{24\pi G(1+w)}\phi}. \tag{7}$$

Thus,

$$\frac{H}{\dot{\phi}} = \sqrt{\frac{8\pi G}{3(w+1)}}. \tag{8}$$

1.2 Perturbed equation of motion in FLRW

The perturbed scalar field equation in the FLRW background without considering the accompanying metric perturbations is

$$\delta\ddot{\phi} + 3H\delta\dot{\phi} + \left(-a^{-2}\nabla^{(3)2} + V_{,\phi\phi}\right)\delta\phi = 0, \tag{9}$$

where $\delta\phi(\mathbf{x}, t)$ is a perturbed part of the scalar field, $\mathbf{x}$ is the comoving spatial coordinate, and $\nabla^{(3)2}$ is the background comoving three-space Laplacian. If we do not consider fluctuations in the metric, the perturbed equation can be derived from the one based on quantum field in curved spacetime [16]. In FLRW background, the scalar field equation $\Box\phi + V_{,\phi} = 0$ leads to

$$\ddot{\phi} + 3H\dot{\phi} - a^{-2}\nabla^{(3)2}\phi + V_{,\phi} = 0. \tag{10}$$

where $\phi = \phi(\mathbf{x}, t)$. In perturbative approach, considering $\phi(\mathbf{x}, t) = \bar{\phi}(t) + \delta\phi(\mathbf{x}, t)$ we can derive Eq. (6) for the background and Eq. (9) for the perturbed part. Notice that unless we introduce a self interacting potential Eq. (9) has the same structure as Eq. (10).

2 Perturbed Scalar Field in the Uniform-curvature Gauge

Gauge condition: The most general scalar-type perturbation of the FLRW spacetime is written as

$$ds^2 = -(1+2\alpha)\,dt^2 - \chi_{,\alpha}dtdx^\alpha + a^2\delta_{\alpha\beta}(1+2\varphi)\,dx^\alpha dx^\beta. \tag{11}$$

Without losing generality we choose a spatial gauge condition which completely fixes the spatial gauge mode; see §3 of [10]. In this paper we ignore the rotation and the gravitational wave modes which are completely decoupled in the FLRW background. For the scalar field we let $\phi(\mathbf{x}, t) = \phi(t) + \delta\phi(\mathbf{x}, t)$. The perturbed order quantities $\alpha(\mathbf{x}, t)$, $\varphi(\mathbf{x}, t)$, $\chi(\mathbf{x}, t)$, and $\delta\phi(\mathbf{x}, t)$ are spatially gauge-invariant, but are temporally gauge dependent; letting any one of these variables equal to zero can be used as the temporal gauge condition. The uniform-curvature gauge condition imposes

$$\varphi(\mathbf{x}, t) \equiv 0. \tag{12}$$

In this gauge the perturbed part of the three-space curvature (intrinsic curvature) of a chosen spacelike hypersurface vanishes, thus justifies its name; see Eq. (11). In the uniform-curvature gauge, variables $\alpha(\mathbf{x}, t)$ and $\chi(\mathbf{x}, t)$ completely characterize the scalar type metric fluctuations; they represent the perturbed lapse function (time-time component of the metric) and the shear variable of the hypersurface, respectively. Using the gauge transformation properties of the variables in §2.2 of [13] we can construct the following gauge invariant combinations:

$$\delta\phi_\varphi \equiv \delta\phi - \frac{\dot{\phi}}{H}\varphi, \quad \alpha_\varphi \equiv \alpha - \left(\frac{\varphi}{H}\right)^{\cdot}, \quad \chi_\varphi \equiv \chi - \frac{\varphi}{H}. \tag{13}$$

In the uniform-curvature gauge we have $\delta\phi_\varphi = \delta\phi$, etc. Thus, the variables in the uniform-curvature gauge can be equivalently considered as the corresponding gauge invariant combinations of the variables with φ. This is true as long as the chosen gauge condition fixes the gauge mode completely [13]. Thus, in the uniform-curvature gauge, we have

$$\delta\phi = \delta\phi_\varphi, \quad \alpha = \alpha_\varphi, \quad \chi = \chi_\varphi. \tag{14}$$

Equation: Since the uniform-curvature gauge condition completely removes the gauge-mode the equation can be managed into a second order differential equation which contains only the physical modes. For a derivation, we can use Eqs. (88) and (89) of [13]; in that paper we set $8\pi G \equiv 1$ and equations are presented for a multi-component case, thus change $\delta\chi_{(i)}$ and $\chi_{(i)}$ into $\delta\phi(\mathbf{x}, t)$ and $\phi(t)$, respectively. (A straight derivation from the Einstein' equation can be found in [10].) Equation (88) of [13] is $(k^2 \rightarrow -\nabla^{(3)2})$

$$\delta\ddot{\phi}_\varphi + 3H\delta\dot{\phi}_\varphi + \left(-a^{-2}\nabla^{(3)2} + V_{,\phi\phi}\right)\delta\phi_\varphi = \dot{\phi}\left(\dot{\alpha}_\varphi - a^{-2}\nabla^{(3)2}\chi_\varphi\right) - 2V_{,\phi}\alpha_\varphi. \tag{15}$$

Equation (15) is a perturbed part of the equation of motion for the scalar field. The terms on the right hand side can be considered as contributions from the perturbed metric; for interpretations see §5 of [10]. Since these metric perturbations are caused by the presence of the perturbed scalar field, there should exist relations between these two perturbations. The relations are provided by Einstein's equation as follows. First two of Eq. (89) in [13] are $(\varphi \equiv 0, K \equiv 0)$:

$$-a^{-2}H\nabla^{(3)2}\chi_\varphi + \left(3H^2 - 4\pi G\dot{\phi}^2\right)\alpha_\varphi = 4\pi G\left(\dot{\phi}\delta\dot{\phi}_\varphi + V_{,\phi}\delta\phi_\varphi\right), \tag{16}$$

$$H\alpha_\varphi = 4\pi G\dot{\phi}\delta\phi_\varphi. \tag{17}$$

Using Eqs. (16) and (17) the right hand side of Eq. (15) can be replaced by $\delta\phi_\varphi$, and we have

$$\delta\ddot{\phi}_\varphi + 3H\delta\dot{\phi}_\varphi + \left[-a^{-2}\nabla^{(3)2} + V_{,\phi\phi} + 2\frac{\dot{H}}{H}\left(3H - \frac{\dot{H}}{H} + 2\frac{\ddot{\phi}}{\dot{\phi}}\right)\right]\delta\phi_\varphi = 0. \tag{18}$$

This is a perturbed scalar field equation which is generally valid in a flat FLRW background dominated by the background scalar field. Compared with Eq. (9), the additional terms in Eq. (18) came from the metric perturbations in Eq. (15). Equations in other gauge choices are more complicated; for a complete presentation see [12].

Limiting cases: Equation (18) has the following interesting consequences. In the small scale, Eq. (18) approaches the same limit of Eq. (9); i.e., the spatial gradient term dominates over other terms. This also happens in many other gauge choices. Thus, WKB approximation solutions agree in either case. Also, in the limit of $G \rightarrow 0$ we recover Eq. (9). In the exponential expansion case, where H is constant, Eq. (18) becomes Eq. (9). (In an *exact* de Sitter background case, from the fact that ϕ is constant both in space and time (§1.1), it follows that $\delta\phi$ is gauge-invariant; see Eq. (19) of [13]. Thus, since the case of uniform-curvature gauge does, $\delta\phi(\mathbf{x}, t)$ equation in any other gauge choice should reduce to Eq. (9) without potential term; this may be one point why the previous analysis in different gauges approximately managed the rigorous result in a near de Sitter space. We can show that even in the power law expansion case, which is supported by the background scalar field, a nontrivial cancellation occurs so that Eq. (18) reduces to Eq. (9) without the potential term. For a general potential, from Eqs. (4), (5), and (6) we can show

$$V_{,\phi\phi} + 2\frac{\dot{H}}{H}\left(3H - \frac{\dot{H}}{H} + 2\frac{\ddot{\phi}}{\dot{\phi}}\right) = \frac{H}{a^3\dot{\phi}}\left[\frac{a^3\dot{\phi}^2}{H^2}\left(\frac{H}{\dot{\phi}}\right)^{\cdot}\right]^{\cdot}. \tag{19}$$

Thus, in the power-law case, using Eq. (8), Eq. (19) vanishes. The general condition for such a cancellation to occur is

$$a^3\left(\dot{\phi}/H\right)^{\cdot} = \text{constant}, \tag{20}$$

which contains the power-law expansion with Eq. (8) as a subset; since $\dot{\phi}$ goes to zero, the exponential expansion is a subset of the power-law case.

General asymptotic solutions: For general $V(\phi)$, the large scale asymptotic solution can be obtained from Eq. (18). We can arrange Eq. (18) into a compact form as

$$\frac{H}{a^3\dot{\phi}}\left[\frac{a^3\dot{\phi}^2}{H^2}\left(\frac{H}{\dot{\phi}}\delta\phi_\varphi\right)^{\cdot}\right]^{\cdot} - a^{-2}\nabla^{(3)2}\delta\phi_\varphi = 0. \tag{21}$$

Neglecting the spatial gradient term we have the large scale integral form solution

$$\delta\phi_\varphi(\mathbf{x}, t) = \frac{\dot{\phi}}{H}\left[-C(\mathbf{x}) + D(\mathbf{x})\int_0^t \frac{H^2}{a^3\dot{\phi}^2}dt\right], \tag{22}$$

where $C(\mathbf{x})$ and $D(\mathbf{x})$ are the coefficients of the growing and the decaying modes, respectively. Notice that $C(\mathbf{x})$ and $D(\mathbf{x})$ remain constant independent of the changing background equation of state, i.e., for

general $V(\phi)$. It is generally possible in other gauge choices to derive the corresponding general large scale asymptotic solution; see Table 1 of [12]. However, a similar general large scale solution is not available in Eq. (9), unless, e.g., the potential term vanishes.

The author of [7] introduced a gauge invariant combination v which is the same as our $a\delta\phi_\varphi$, thus $a\delta\phi$ in the uniform-curvature gauge. (Although the author of [7] introduced v, which is $a\delta\phi$ in the uniform-curvature gauge, the advantage of using v as a uniform-curvature gauge variable was not realized in [7, 8].) In terms of v Eq. (18) can be written as

$$v'' + \left(k^2 - \frac{z''}{z}\right)v = 0, \quad z \equiv \frac{a\dot{\phi}}{H}, \tag{23}$$

where a prime denotes the time derivative based on the conformal time η, $d\eta = a^{-1}dt$. In the large scale limit $(z''/z \gg k^2)$ we have $v(\mathbf{x}, \eta) = c_g(\mathbf{x})z + c_d(\mathbf{x})z\int_0^\eta d\eta/z^2$. By matching $c_g = -C$ and $c_d = D$, it is equivalent to Eq. (22). In the small scale limit $(z''/z \ll k^2)$ we have $v = ce^{ik\eta} + de^{-ik\eta}$, thus

$$\delta\phi_\varphi(\mathbf{k}, \eta) = \frac{1}{a}\left[c(\mathbf{k})e^{ik\eta} + d(\mathbf{k})e^{-ik\eta}\right], \tag{24}$$

where $c(\mathbf{k})$ and $d(\mathbf{k})$ are constant coefficients.

3 Quantum Generation Stage

In the exponential and the power law expansion stages we showed that Eq. (18) becomes Eq. (9), thus with Eq. (10) without self-interaction. That is, in the exponential and the power law expansion backgrounds the contribution from the metric fluctuations in Eq. (18) disappears. Thus, in these expansion stages we can adapt the previous (quantum field in curved spacetime) efforts invested in managing Eq. (10) into our case of Eq. (18).

In our perturbative semiclassical approach, we replace $\delta\phi_\varphi(\mathbf{x}, t)$ with a quantum (Heisenberg representation) operator $\delta\hat{\phi}_\varphi(\mathbf{x}, t)$; $\delta\hat{\phi}_\varphi \equiv \delta\hat{\phi} - (\dot{\phi}/H)\hat{\varphi}$. Considering that we are in a flat background we have a mode expansion as

$$\delta\hat{\phi}_\varphi(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^{3/2}}\left[\hat{a}_\mathbf{k}\delta\phi_{\varphi\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_\mathbf{k}^\dagger\delta\phi_{\varphi\mathbf{k}}^*(t)e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \tag{25}$$

where $\hat{a}_\mathbf{k}$ and $\hat{a}_\mathbf{k}^\dagger$ satisfy the standard commutation relation; $[\hat{a}_\mathbf{k}, \hat{a}_{\mathbf{k}'}^\dagger] = \delta^3(\mathbf{k} - \mathbf{k}')$ and 0, otherwise. The corresponding perturbed metric variables are similarly changed into operators. Using Eqs. (16) and (17) the operator forms of the metric variables ($\hat{\alpha}_\varphi$ and $\hat{\chi}_\varphi$) can be expressed as linear combinations of $\delta\hat{\phi}_\varphi$. Equation (18) yields

$$\delta\ddot{\phi}_{\varphi\mathbf{k}} + 3H\delta\dot{\phi}_{\varphi\mathbf{k}} + \left[\frac{k^2}{a^2} + V_{,\phi\phi} + 2\frac{\dot{H}}{H}\left(3H - \frac{\dot{H}}{H} + 2\frac{\ddot{\phi}}{\dot{\phi}}\right)\right]\delta\phi_{\varphi\mathbf{k}} = 0. \tag{26}$$

In order to have a proper normalization of the quantum fluctuations, we need the correct equal time commutation relation. For this we may need a Lagrangian formulation of scalar field perturbation analysis considering the accompanying metric fluctuations. Such analysis is available in the literature [7, 8]. From the Lagrangian formulation (presented by using v) in [7] we can derive the corresponding action in terms of $\delta\phi_\varphi$ as

$$S = \frac{1}{2}\int a^3\sqrt{g^{(3)}}\left\{\delta\dot{\phi}_\varphi^2 - \frac{1}{a^2}\delta\phi_{\varphi,\alpha}\delta\phi_\varphi^{|\alpha} - \frac{H}{a^3\dot{\phi}}\left[\frac{a^3\dot{\phi}^2}{H^2}\left(\frac{H}{\dot{\phi}}\right)^{\cdot}\right]^{\cdot}\delta\phi_\varphi^2\right\}dtd^3x, \tag{27}$$

where $g^{(3)} = 1$ in a flat background, and a vertical bar indicates the covariant derivative based on $g_{\alpha\beta}^{(3)}$ (which is $\delta_{\alpha\beta}$ in our flat background). Considering $S = \int \mathcal{L}dtd^3x$, the conjugate momenta is derived as $\delta\pi_\varphi \equiv \partial\mathcal{L}/(\partial\delta\dot{\phi}_\varphi) = a^3\delta\dot{\phi}_\varphi$. Thus, from the equal time commutation relation $[\delta\hat{\phi}_\varphi(\mathbf{x}, t), \delta\hat{\pi}_\varphi(\mathbf{x}', t)] = i\delta^3(\mathbf{x} - \mathbf{x}')$ we can derive

$$[\delta\hat{\phi}_\varphi(\mathbf{x}, t), \delta\dot{\hat{\phi}}_\varphi(\mathbf{x}', t)] = ia^{-3}\delta^3(\mathbf{x} - \mathbf{x}'). \tag{28}$$

The power spectrum is defined as

$$\mathcal{P}_{\delta\hat{\phi}}(k, t) \equiv [\Delta\hat{\phi}(k, t)]^2 \equiv \frac{k^3}{2\pi^2}|\delta\phi_\mathbf{k}(t)|^2 = \frac{k^3}{2\pi^2}\int\langle\delta\hat{\phi}(\mathbf{x} + \mathbf{r}, t)\delta\hat{\phi}(\mathbf{x}, t)\rangle_{\text{vac}}e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r, \tag{29}$$

where $\langle\rangle_{\text{vac}} = \langle\text{vac}||\text{vac}\rangle$ indicates a vacuum expectation value with $\hat{a}_{\mathbf{k}}|\text{vac}\rangle \equiv 0$.

Eq. (26) can be solved in the following two background evolution stages. In both cases Eq. (26) reduces to Eq. (9) without the potential term

$$\delta\ddot{\phi}_{\varphi\mathbf{k}} + 3H\delta\dot{\phi}_{\varphi\mathbf{k}} + \frac{k^2}{a^2}\delta\phi_{\varphi\mathbf{k}} = 0. \tag{30}$$

In an *exponential expansion* case Eq. (30) has an exact solution [17, 18, 9]

$$\delta\phi_{\varphi\mathbf{k}}(t) = c_1(k)\frac{1}{\sqrt{2ka}}\left(1 + \frac{i}{k\eta}\right)e^{ik\eta} + c_2(k)\frac{1}{\sqrt{2ka}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}. \tag{31}$$

The coefficients $c_1(k)$ and $c_2(k)$ are arbitrary functions of k which are normalized as

$$|c_2(k)|^2 - |c_1(k)|^2 = 1, \tag{32}$$

according to Eq. (28). Thus $c_1(k)$ and $c_2(k)$ are not completely fixed. The different choices of $c_1(k)$ and $c_2(k)$ can be interpreted as the different choices for the vacuum state. An adiabatic vacuum (in de Sitter space it is often called as Bunch-Davies vacuum [17]) corresponds to choosing

$$c_2(k) \equiv 1, \quad c_1(k) \equiv 0. \tag{33}$$

In the large scale, but in a general vacuum, Eq. (29) becomes

$$\mathcal{P}^{1/2}_{\delta\hat{\phi}_\varphi}(k,t) = \frac{H}{2\pi}\left|c_2(k) - c_1(k)\right|. \tag{34}$$

In a *power law expansion* stage, with $\mathrm{w} < -\frac{1}{3}$, we can derive the exact solution of Eq. (30) as [19, 20, 9] $[a \equiv (\eta/\eta_0)^{2/(1+3\mathrm{w})} a_0]$

$$\delta\phi_{\varphi\mathbf{k}}(t) = -\frac{\sqrt{\pi\eta}}{2a}\left[c_1(k)H_\nu^{(1)}(k\eta) + c_2(k)H_\nu^{(2)}(k\eta)\right], \quad \nu \equiv \frac{3(\mathrm{w}-1)}{2(3\mathrm{w}+1)}. \tag{35}$$

In the exponential expansion limit we have $\mathrm{w} = -1$ ($\nu = \frac{3}{2}$) and Eq. (35) approaches Eq. (31). The coefficients $c_1(k)$ and $c_2(k)$ are normalized according to Eq. (32). In the large scale, Eq. (29) becomes

$$\mathcal{P}^{1/2}_{\delta\hat{\phi}_\varphi}(k,t) = \frac{1}{\pi^{3/2}a|\eta|}\Gamma\left[\frac{3(\mathrm{w}-1)}{2(3\mathrm{w}+1)}\right]\left(\frac{k|\eta|}{2}\right)^{3(\mathrm{w}+1)/(3\mathrm{w}+1)}\left|c_2(k) - c_1(k)\right|, \tag{36}$$

Since $a \propto \eta^{1/2-\nu}$, Eq. (36) does not depend on any epoch and remains constant in the large scale. In the exponential expansion limit, we have $\mathrm{w} = -1$, and Eq. (36) approaches Eq. (34). An adiabatic vacuum corresponds to choosing $c_2(k) \equiv 1$, $c_1(k) \equiv 0$.

4 Classical Evolution Stage

We consider a later stage of perturbed FLRW model filled with an ideal fluid. The evolution equation for density fluctuations in the comoving gauge is well known. From Eq. (41) of [21] we have

$$\ddot{\delta}_\Psi + \left(2 - 6\mathrm{w} + 3c_s^2\right)H\dot{\delta}_\Psi - \left[\frac{c_s^2}{a^2}\nabla^{(3)2} + 4\pi G\mu\left(1 - 6c_s^2 + 8\mathrm{w} - 3\mathrm{w}^2\right)\right]\delta_\Psi = 0, \tag{37}$$

where $\delta \equiv \delta\mu/\mu$, $\mathrm{w} \equiv p/\mu$, and $c_s^2 \equiv \dot{p}/\dot{\mu}$; we neglect the imperfect fluid contributions and assume $K = 0 = \Lambda$. The comoving gauge fixes the temporal gauge mode completely. $\delta_\Psi \equiv \delta - 3H\Psi$ is a gauge invariant combination, and becomes δ in the comoving gauge which sets $\Psi \equiv 0$. Ψ is a velocity related variables, see Eqs. (8) and (44) of [13]. In the Newtonian limit ($p \to 0$) Eq. (37) reduces to the corresponding Newtonian perturbation equation. With this as one reason we suggested that δ in the comoving gauge most closely resembles the Newtonian mass density fluctuation ($\delta\varrho/\varrho$) [14].

In the large scale we can derive a general integral form solution [see Eq. (47) of [21]]

$$\delta_\Psi(\mathbf{x},t) = -\frac{2}{3}\nabla^{(3)2}\left[\frac{1}{\dot{a}^2}C(\mathbf{x})\left(1 - \frac{H}{a}\int_0^t a\,dt\right) + \frac{1}{a^2\dot{a}}d(\mathbf{x})\right], \tag{38}$$

where $C(\mathbf{x})$ and $d(\mathbf{x})$ are coefficients for the growing and the decaying modes, respectively. This solution is valid in considering the general time varying $p = p(\mu)$. We matched the solutions so that $C(\mathbf{x})$ is the

same as the one in Eq. (22); all solutions are linearly interconnected with each other. The complete large scale solutions in various gauge are presented in Table 8 of [15] for an ideal fluid, and in Table 1 of [12] for a scalar field. From Table 1 of [12] we find that $D(\mathbf{x})$ in Eq. (22) is $k^2/(aH)^2$ order higher than $d(\mathbf{x})$. This means that the dominating decaying mode for $\delta\phi$ in the uniform-curvature gauge vanishes.

In Newtonian gravity the Poisson's equation relates the potential with the density field. In relativistic perturbation analyses a perturbed potential (curvature) variable, φ, in the zero-shear gauge is related to the perturbed density variable in the comoving gauge through Poisson-like equation; see §3.8 of [21]. (The zero-shear gauge condition also fixes the temporal gauge mode completely. $\varphi_\chi \equiv \varphi - H\chi$ is the corresponding gauge invariant combination, and becomes φ in the zero-shear gauge which fixes $\chi \equiv 0$.) Thus, from $-a^{-2}\nabla^{(3)2}\varphi_\chi = 4\pi G\delta\mu_\Psi$ we have

$$\varphi_\chi(\mathbf{x}, t) = C(\mathbf{x})\left(1 - \frac{H}{a}\int_0^t a\,dt\right) + \frac{H}{a}d(\mathbf{x}). \tag{39}$$

Physically, we can understand this connection between $\delta\phi_\varphi$ in the early universe and δ_Ψ in later era as follows. During the scalar field dominated era, when the scale we are considering was microscopic, quantum fluctuations in the field ($\delta\hat{\phi}$) will simultaneously excite quantum fluctuations in the metric field (e.g., $\hat\alpha$, $\hat\chi$, or $\hat\varphi$ in other than the uniform-curvature gauge). The uniform-curvature gauge is suitable for treating the scalar field fluctuation. Due to the accelerated expansion the scale soon becomes macroscopic, and both fluctuations may become classical. For a coherent picture, there must be a transition from the scalar field dominated stage to the fluid dominated stage which is often termed as reheating. During such transition stage the scale we are considering was far larger than horizon and the details of the reheating process will not affect the fluctuations in such a scale. While the scale remains in the large scale regime we can consider the information about fluctuation is coded in the metric fluctuations whose growing mode is characterized by $C(\mathbf{x})$. Remember that $C(\mathbf{x})$ is a constant coefficient for solutions which include the evolving $p(\mu)$ or $V(\phi)$ in the large scale limit. During the fluid dominated era the generated fluctuations in the metric (φ) will simultaneously reside together with corresponding fluctuations in the fluid quantities. In this way the fluctuation in δ in fluid era [Eq. (38)] is related to the fluctuations in the scalar field in the early scalar field dominated era [Eq. (22)].

For w = constant, we can derive an exact solution of Eq. (37) which is valid for general scale; see §3.2 of [15]. In Fourier space, we have

$$\delta_{\Psi\mathbf{k}}(t) = x^{2-\beta}\left[\hat{a}j_\beta(y) + \hat{b}n_\beta(y)\right], \quad y \equiv \sqrt{\mathrm{w}}x, \quad x \equiv \beta\frac{k}{\dot{a}}, \quad \beta \equiv \frac{2}{1+3\mathrm{w}}, \tag{40}$$

where $\hat{a}(\mathbf{k})$ and $\hat{b}(\mathbf{k})$ are constant coefficients. In the radiation dominated era, since w $= \frac{1}{3}$, Eq. (40) becomes

$$\delta_{\Psi\mathbf{k}}(t) = \sqrt{3}\left[\hat{a}\left(\frac{\sin y}{y} - \cos y\right) - \hat{b}\left(\frac{\cos y}{y} + \sin y\right)\right]. \tag{41}$$

In this case Eq. (38) becomes

$$\delta_\Psi(\mathbf{x}, t) = -\frac{2(1+\mathrm{w})}{5+3\mathrm{w}}\frac{1}{\dot{a}^2}\nabla^{(3)2}C - \frac{2}{3}\frac{1}{a^2\dot{a}}\nabla^{(3)2}d \quad \propto \quad t^{\frac{2}{3}\frac{1+3\mathrm{w}}{1+\mathrm{w}}}C, \quad t^{\frac{1-\mathrm{w}}{1+\mathrm{w}}}d. \tag{42}$$

In the matter dominated era, the spatial gradient term in Eq. (37) is negligible as long as the scale we are considering is larger than Jeans scale (sound horizon) which is negligible compared to the visual horizon; for the Newtonian analyses see §3.3 of [21]. Thus, in the matter dominated era the solution in Eq. (38) remains valid in the regime larger than Jeans scale. If the scale came inside the visual horizon during the matter dominated era we have

$$\delta_{\Psi\mathbf{k}}(t) = \frac{2}{5}\frac{k^2}{\dot{a}^2}C_\mathbf{k} + \frac{2}{3}\frac{k^2}{a^2\dot{a}}d_\mathbf{k} \quad \propto \quad t^{\frac{2}{3}}C_\mathbf{k}, \quad t^{-1}d_\mathbf{k}. \tag{43}$$

Thus, at the (second) horizon crossing epoch in the matter dominated era, thus $k/(aH)|_{\mathrm{HC}} = 1$, neglecting the decaying mode, we have

$$\delta_\Psi|_{\mathrm{HC}} = \frac{2}{5}C. \tag{44}$$

In the radiation dominated era the spatial gradient term in Eq. (37) cannot be neglected near horizon crossing; in the radiation dominated era the sound horizon scale is comparable to the visual horizon with $c_s = 1/\sqrt{3}$. Thus, solution in Eq. (38) is not valid near and after the horizon crossing due to the violation

of the large scale condition. In the radiation dominated era density fluctuations inside horizon start to oscillate; see Eq. (41). However, its amplitude remains constant. Considering only the growing mode, from Eqs. (38) and (41) we can match the coefficients for the growing modes. (The same results are obtained by using continuous variables in sudden jump approximation of transitions between different background equation of states [22].) Thus, in the small scale limit (SS) in the radiation dominated era the amplitude of δ becomes

$$\delta_\Psi|_{\mathrm{SS,AMP}} = 4C. \tag{45}$$

In this section we reviewed some facts necessary for our proof of the inflation generated density spectrums which will be presented in next section. Concerning the classical evolution stage one can find another way of presentation in §IV of [9].

5   Inflationary Spectrum

An ansatz: Analogous to our definition of power spectrum in the quantum context [Eq. (29)], we can similarly introduce the classical (notice no hat) power spectrum based on a classical fluctuating field, $f(\mathbf{x}, t)$, as

$$\mathcal{P}_f(k,t) \equiv [\Delta f(k,t)]^2 \equiv \frac{k^3}{2\pi^2} |f_\mathbf{k}(t)|^2 = \frac{k^3}{2\pi^2} \int \langle f(\mathbf{x}+\mathbf{r},t) f(\mathbf{x},t) \rangle_\mathbf{x} e^{-i\mathbf{k}\cdot\mathbf{r}} d^3r, \tag{46}$$

where $f_\mathbf{k}(t)$ is a Fourier transform of $f(\mathbf{x}, t)$; $\langle\rangle_\mathbf{x}$ is a spatial averaging. This explains why we need the absolute value in Eq. (1). An *ansatz* commonly adopted in the literature is to identify

$$\mathcal{P}_{\delta\phi}(k,t) \equiv \mathcal{P}_{\delta\hat\phi}(k,t). \tag{47}$$

In order to distinguish this ansatz we keep hats in Eqs. (1) and (2). The power spectrum of the density fluctuations in Eq. (46) is closely related to the mass fluctuations; see Eq. (69) of [9]. In this context, $\mathcal{P}_\delta(k, t)$ is often used in characterizing density fluctuations.

Generated spectrum: The power spectrums, $\Delta\hat\phi_\varphi(k, t)$ for the exponential inflation and the power law inflation are derived in Eqs. (34) and (36), respectively. Both expressions remain constant in time. By using an adiabatic vacuum, Eq. (34) becomes Eq. (2) for an exponential expansion.

Although they are evaluated in different eras and different gauges, the same $C(\mathbf{x})$ appears in Eqs. (22), (44), and (45). This reflects the fact that through linear evolution we do not have structure formation; information about the spatial structure is coded in $C(\mathbf{x})$ and the coefficient for the decaying mode. Combine Eqs. (22), (44), and (45). Replace $\delta_{\Psi\mathbf{k}}(t)$ and $\delta\phi_{\varphi\mathbf{k}}(t)$ with the quantities characterizing the fluctuations in Eqs. (29) and (46). Then, using an ansatz in Eq. (47), we can derive Eq. (1). This completes our argument presented in the introduction.

Equation (3) is based on a widely accepted vacuum choice; see below Eq. (32). In fact, we can write it in an arbitrary vacuum. The corresponding density spectrum for the exponential and the power law inflation can be found from Eqs. (1), (34), and (36):

$$\left.\frac{\Delta\mu}{\mu}\right|_{\mathrm{MDE,HC}} = \left| -\frac{1}{5\pi}\frac{H^2}{\dot\phi} \right| \times \left| c_2(k) - c_1(k) \right|\Big|_{\mathrm{EXP}}, \tag{48}$$

$$\left.\frac{\Delta\mu}{\mu}\right|_{\mathrm{MDE,HC}} = \left| -\sqrt{\frac{8\pi G}{3(\mathrm{w}+1)}}\frac{H(3\mathrm{w}+1)}{5\pi^{3/2}}\Gamma\left[\frac{3(\mathrm{w}-1)}{2(3\mathrm{w}+1)}\right]\left(\frac{k|\eta|}{2}\right)^{3(\mathrm{w}+1)/(3\mathrm{w}+1)} \right| \left| c_2(k) - c_1(k) \right|\Big|_{\mathrm{POW}} \tag{49}$$

where $c_1(k)$ and $c_2(k)$ only subject to a condition in Eq. (32). Corresponding results can be given in the radiation dominated era case. We used the initial condition which is generated from quantum fluctuations evaluated in the large scale; Eqs. (34) and (36). If we put the constraint that in the small scale the solution should have corresponded to the positive frequency adiabatic vacuum [see below Eq. (32)], we recover Eq. (3) in the exponential case. It is often accepted that the observable part of the universe, thus the observed large scale structure in it, may come out from the range where the adiabatic vacuum choice is relevant [19, 18].

From Eqs. (46) and (43) we have

$$\left.\frac{\Delta\mu}{\mu}\right|_{\mathrm{HC}} = \frac{\dot a^2}{\sqrt{2}\pi}k^{-1/2}|\delta_\mathbf{k}(t)|. \tag{50}$$

Following the convention $|\delta_{\mathbf{k}}(t)|^2 \equiv A(t)k^n$, Eqs. (48) and (49) in the adiabatic vacuum situation lead to

$$n|_{\mathrm{EXP}} = 1, \quad n|_{\mathrm{POW}} = \frac{9\mathrm{w} + 7}{3\mathrm{w} + 1}. \tag{51}$$

The first one is known as Zel'dovich spectrum; $\ln k$ dependence may still arise as we evaluate the time varying $H^2/\dot{\phi}$ (especially $1/\dot{\phi}$) in specific inflation models which may deviate (not much of the expansion dynamics, but the physical state of $\dot{\phi}$; see §1.1) from the exact de Sitter case; in an exact exponential expansion limit Eq. (3) diverges. The spectrum for the power law case in Eq. (51) is the same as the one in [23].

6 Generalized Gravity Theories

Using the uniform-curvature gauge a class of generalized gravity theories also yields the simple form of the large scale solution. The following Lagrangian includes various types of generalized gravity theories [24]

$$\mathcal{L} = \sqrt{-g} \left[\frac{1}{2} f(\phi, R) - \frac{1}{2}\omega(\phi)\phi^{;a}\phi_{,a} - V(\phi) \right]. \tag{52}$$

It includes $f(R)$ gravity, scalar-tensor theory, nonminimally coupled scalar field, induced gravity, etc, as subclasses. The general theory included in this Lagrangian is a two component system. Thus, in general, we may end up with a fourth order differential equation which describes the scalar type perturbations. However, each subclass we mentioned corresponds to one component system. The perturbation analysis can be made in a unified manner similarly as in the case of minimally coupled scalar field [24, 13, 8]. Part of the reason why we have such a simple treatment is that there exists a conformal transformation which transforms each case of generalized gravity theory into Einstein's gravity supplemented by a minimally coupled scalar field with a special potential. For a perturbation treatment, however, we can proceed it without using the conformal transformation. The analyses were previously made based on the zero-shear gauge [24, 13]. In the following we present a translation of the solutions derived in the zero-shear gauge into ones in the uniform-curvature gauge [25].

In the zero-shear gauge we derived the large scale integral form solutions which are valid for general $V(\phi)$ [see Eqs. (85) and (86) of [13]]:

$$\frac{\delta\phi_\chi}{\dot{\phi}} = \frac{\delta F_\chi}{\dot{F}} = -C\frac{1}{aF}\int_0^t aFdt + \frac{1}{aF}d, \quad \varphi_\chi = C\left(1 - \frac{H}{aF}\int_0^t aFdt\right) + \frac{H}{aF}d, \tag{53}$$

where $F \equiv f_{,R}$. The coefficients $C(\mathbf{x})$ and $d(\mathbf{x})$ are matched to the solutions in the minimally coupled scalar field and the ideal fluid; see Eqs. (22) and (39). $\delta\phi$ in the uniform-curvature gauge is the same as $\delta\phi_\varphi$ in Eq. (13). Evaluating $\delta\phi_\varphi$ in the zero-shear gauge and using the solutions in Eq. (53) we can derive the large scale solutions in the uniform-curvature gauge [we have $\delta F_\varphi \equiv \delta F - (\dot{F}/H)\varphi$]:

$$\delta\phi_\varphi = -\frac{\dot{\phi}}{H}C(\mathbf{x}), \quad \delta F_\varphi = -\frac{\dot{F}}{H}C(\mathbf{x}). \tag{54}$$

Disappearence of the decaying mode in the uniform-curvature gauge means that the dominating decaying mode vanishes. The decaying mode in Eq. (22) is higher order (in large scale expansion) compared with the one in other gauge; see below Eq. (37) and [12]. $\delta\phi_\varphi$ in Eq. (54) is simpler than $\delta\phi_\chi$ in Eq. (53). Neglecting the decaying mode, even in the generalized gravity theories we have

$$C(\mathbf{x}) = -\frac{H}{\dot{\phi}}\delta\phi_\varphi(\mathbf{x}, t), \tag{55}$$

which is free of F and ω, thus, is the same as Eq. (22) valid for the minimally coupled scalar field.

The solutions in the uniform-curvature gauge above were obtained by using the known solutions in the zero-shear gauge. Using the complete set of equations presented in §4.2 of [13] we can directly derive the equation for $\delta\phi_\varphi$ and its asymptotic solutions in the generalized $f(\phi, R)$ gravity; recently, we present a thorough analyses in [27]. Whether the uniform-curvature gauge could allow the analytic derivation of quantum fluctuations in the context of nonminimally coupled scalar field or other generalized gravity theories is an interesting question. We would like to address this subject in a future occasion.

7 Summary

We summarize the result in the following.

- Perturbative semiclassical approximation simultaneously treats both the perturbed field and the perturbed metric quantum mechanically. By appropriate choice of the gauge, the perturbative approach is easily tractable and becomes more self-consistent than the quantum field analysis in curved spacetime.

- We show the uniform-curvature gauge provides the most convenient analysis in treating the scalar field and a class of generalized gravity theories.

- In the large scale, the equation in the uniform-curvature gauge yields an integral form solution for a general $V(\phi)$; Eq. (22). This is not possible if we neglect the metric perturbations. Similar large scale integral form solutions are available in the other gauges, but they are more complicated; for a thorough analysis in other gauges, see [12]. The general solution in the small scale limit is presented in Eq. (24).

- The perturbed scalar field equation in the uniform-curvature gauge exactly reduces to the one which neglects the metric perturbations in the exponential and the power law expanding backgrounds. In these cases we consider the background dynamics is supported by the background scalar field. Meanwhile, in the approach of the quantum field theory in curved spacetime the background has been assumed to be given by an external source.

- The quantum initial conditions can be derived in general scale. Thus, we can avoid any small and large scale matching at horizon crossing epoch; §3.

- The previously known and also widely used formulae are derived in exact settings; Eqs. (1), (3), and (49). This enables us to evaluate Newtonian density perturbations in terms of quantum fluctuations generated in an inflation era; Eqs. (3), (48), and (49). Taking a suitable gauge condition is essential for making the above simple analysis possible, particularly in quantum generation processes.

- Our result in Eq. (1) can be generally applied to inflation models based on the minimally coupled scalar field. Notice that in Eq. (1), $V(\phi)$ is general as long as the dynamics is governed by the scalar field.

- The density spectra in Eq. (51), are not only characterized by the background expansion dynamics at the quantum generation stage, but also depend on the choice of the natural vacuum state; Eqs. (48) and (49).

- We used a semiclassical method in quantizing only the perturbed parts. In a related context Eq. (47) remains as an ansatz. An important related issue of the classicalization of the quantum fluctuations is not addressed in this paper.

- We presented the simple forms of the large scale integral solutions valid in a class of generalized gravity theories in the uniform-curvature gauge; Eq. (54). Remarkably, considering the growing mode, the same solution for minimally coupled scalar field in Eq. (22) remains valid in a class of generalized gravity theories in Eq. (52).

- Equation (18) is valid for a general scalar field dominated model. Thus, it will be interesting to apply it to situations where the terms appended through the metric fluctuations do not vanish. This is left as a future study.

One lesson we notice from this study is that, although the final physical result should not depend on any gauge choice we are working on, in managing an actual situation, it is very important to choose an appropriate gauge which suits the problem. Indeed, the gauge choice is a freedom we have. In this regards, the gauge ready method proposed in [26, 13] is convenient because it allows an easy adaptation of the perturbed system into any gauge choice and also allows an easy translation between solutions in different gauge choices. It is known that the density fluctuations in the comoving gauge ($\delta\mu|_{\rm CG}$), the potential (or curvature) and the velocity fluctuations in the zero-shear gauge ($\varphi|_{\rm ZSG}$ and $\delta v|_{\rm ZSG}$) most closely resemble the Newtonian counterparts of density, potential, and velocity fluctuations, respectively; see [14]. From our study we may identify one additional case: the scalar field fluctuations in the uniform-curvature gauge ($\delta\phi|_{\rm UCG}$) closely resemble the field fluctuations based on the pure background metric. These fluctuations in the mentioned gauges can be regarded as the gauge-invariant concepts. That is we have $\delta\mu|_{\rm CG} = \delta\mu_\Psi$, $\varphi|_{\rm ZSG} = \varphi_\chi$, $\delta v|_{\rm ZSG} = \delta v_\chi$, and $\delta\phi|_{\rm UCG} = \delta\phi_\varphi$.

Now, we summarize the large scale structure generation and the evolution process. The microscopic quantum fluctuations are expanded into the macroscopic scale during inflation stage and generate the large scale (and macroscopic) fluctuations in the perturbed scalar field. The scalar field dominated stage can be easily handled in the uniform-curvature gauge, §3. The simultaneously excited metric fluctuations can

be characterized by curvature (potential) fluctuations in the fabric of the metric in the zero-shear gauge, Eq. (39). This can be transformed into the Newtonian (the comoving gauge one) density fluctuations later, leading to the results in Eqs. (3), (48), and (49) in some inflation stages we consider. Most of the time the fluctuation scales stay in the large scale (outside horizon for general p, and outside Jeans scale for the matter dominated era). In such scales the evolution of fluctuations is characterized by a conserved quantity $C(\mathbf{x})$ appearing in the large scale solutions. C appears as an integration constant for the growing mode in every variable. The curvature (potential) variable, φ, in many gauge choices is conserved in the large scale and can be identified with C. [φ is C on scales larger than horizon in most of the gauge choices; the exceptions are the zero-shear gauge where we have Eq. (39) and the uniform-curvature gauge where we take $\varphi \equiv 0$ as the gauge condition.] During scalar field dominated inflation era, the spatial information about $\delta\phi(\mathbf{x}, t)$ (generated from vacuum quantum fluctuations) in the uniform-curvature gauge is encoded into $C(\mathbf{x})$, Eq. (22). Ignoring the transient mode we have

$$C(\mathbf{x}) = -\frac{H}{\dot\phi}\delta\phi_\varphi(\mathbf{x}, t), \tag{56}$$

which remains valid even in the generalized gravity theories in Eq. (55). C carries the spatial information about the initial condition. Later the same C can be translated (decoded) into the (growing mode part) information about φ in the zero-shear gauge [Eq. (39)] or δ in the comoving gauge [Eq. (38)] and in fact any variable in any gauge. In an ideal fluid case, $\delta\mu_\Psi/\mu$, $-\varphi_\chi$, v_χ, and $\delta T_\Psi/T$ play the roles of the Newtonian relative density fluctuation ($\delta\varrho/\varrho$), potential fluctuation ($\delta\Phi$), velocity fluctuation (δv), and the relative temperature fluctuations in the cosmic microwave background radiation ($\delta T/T$), respectively [14]. [v is related to Ψ as $v \equiv -(k/a)\Psi/(\mu + p)$. Ψ is a frame invariant velocity related variable; see §2.1.2 of [13].] In the matter dominated stage, ignoring the decaying modes, we have [14] [see Eqs. (43,39)]:

$$\frac{\delta\varrho}{\varrho} = \frac{2}{5}\left(\frac{k}{aH}\right)^2 C, \quad \delta\Phi = -\frac{3}{5}C, \quad \delta v = -\frac{2}{5}\left(\frac{k}{aH}\right)C, \quad \frac{\delta T}{T} = \frac{1}{5}C. \tag{57}$$

Through the linear evolution the spatial structures are preserved. $C(\mathbf{x})$ encodes the spatial structure of the growing mode in the linear regime. We would like to remark that this is *a* way of describing the structure generation process which is simple and concrete.

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